$$\mathrm{TRUE} \triangleq \lambda x.(\lambda y.\ x)$$
$$\mathrm{FALSE} \triangleq \lambda x.\lambda y.\ y$$
$$\mathrm{IF} \triangleq \lambda b.\lambda t.\lambda f.\ b\ t\ f$$

$$\mathrm{IF}\ \mathrm{TRUE}\ x\ y \xrightarrow{*} x$$
$$\mathrm{IF}\ \mathrm{FALSE}\ x\ y \xrightarrow{*} y$$

$$\overline{(\lambda x.\ e)\ v \to e\{x \mapsto v\}}$$

$$(\lambda \underbrace{x}_{x}.\ \underbrace{x\ x}_{e})\ \underbrace{(\lambda x.\ x\ x)}_{v}$$
$$\to\ e\{x \mapsto v\}$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$$

$$e ::= x \mid e_1\ e_2 \mid \lambda x : \tau .\ e \mid n \mid \mathrm{true} \mid \mathrm{false}$$

$$\tau ::= \mathrm{bool} \mid \mathrm{int} \mid \tau_1 \to \tau_2$$

$$\mathrm{bool} \to \mathrm{int}$$
$$(\mathrm{int} \to \mathrm{bool}) \to \mathrm{int}$$

$$4 + \lambda x.x \qquad 4 + 2$$

$$\{y : \mathrm{bool}\} \vdash 4 : \mathrm{int}$$
$$\vdash 4 + 2 : \mathrm{int}$$
$$\vdash \lambda x : \mathrm{int}.\ x : \mathrm{int} \to \mathrm{int}$$
$$\vdash (\lambda x : \mathrm{int}.\ x)\ 4 : \mathrm{int}$$
$$\{x : \mathrm{int}\} \vdash x : \mathrm{int}$$

$$\Gamma : \mathrm{Var} \to \mathrm{Type}$$

$$\vdash \underbrace{\lambda x : \mathrm{int}.\ \underbrace{\underbrace{x}_{\mathrm{int}} + \underbrace{2}_{\mathrm{int}}}_{\mathrm{int}}}_{\mathrm{int} \to \mathrm{int}} : \mathrm{int} \to \mathrm{int}$$

$$\overline{(\lambda x : \tau .\ e)\ v \to e\{x \mapsto v\}}$$

$$\boxed{\Gamma \vdash e : \tau}$$

typecheck(gamma, e) → tau

$$\frac{}{\Gamma \vdash n : \mathrm{int}} \qquad \frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool}}$$

$$\frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool}} \qquad \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}$$

$$\frac{\Gamma[x \mapsto \tau_1] \vdash e : \tau_2}{\Gamma \vdash (\lambda x : \tau_1 .\ e) : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{bool} \quad \Gamma \vdash e_2 : \mathrm{int}}{\Gamma \vdash e_1 + e_2 : \mathrm{int}}$$

$$\frac{\dfrac{\dfrac{\{x \mapsto \mathrm{int}\}(x) = \mathrm{int}}{\{x \mapsto \mathrm{int}\} \vdash x : \mathrm{int}} \quad \dfrac{}{\{x \mapsto \mathrm{int}\} \vdash 38 : \mathrm{int}}}{\dfrac{\{x \mapsto \mathrm{int}\} \vdash x + 38 : \mathrm{int}}{\vdash (\lambda x : \mathrm{int}.\ x + 38) : \mathrm{int} \to \mathrm{int}}} \quad \dfrac{}{\vdash 4 : \mathrm{int}}}{\vdash (\lambda x : \mathrm{bool}.\ x + 38)\ 4 : \mathrm{int}}$$

WELL-TYPE PROGRAMS CANNOT GO WRONG.

<u>SOUNDNESS</u>

$\exists \tau.\ \vdash e : \tau$ $\quad \exists n\ e \xrightarrow{n} e'$

IF $e$ IS WELL-TYPED AND $e \xrightarrow{*} e'$ AND $e'$ IS IRREDUCIBLE, THEN $e'$ IS A VALUE.

$$\nexists e''.\ e' \to e''$$

$$v ::= n \mid \mathrm{true} \mid \mathrm{false} \mid \lambda x : \tau .\ e$$

$$(\lambda x : \mathrm{int}.\ x) + 4$$

$$4 + (1 + 1) \to 4 + 2 \to \underbrace{6}_{v}$$

$$(\lambda x : \mathrm{bool}.\ x)$$

$$\Omega \triangleq (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$$

$$\omega \triangleq \vdash \lambda x : \tau .\ \underbrace{x}_{\tau \to \tau'}\ x : \tau'$$

$$\lambda x : \mathrm{int}.\ x$$

$$\tau = \tau \to \tau'$$

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\ e \mid \#2\ e$$
$$v ::= \cdots \mid (v_1, v_2)$$
$$\tau ::= \cdots \mid \tau_1 \times \tau_2$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\ e : \tau_1}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#2\ e : \tau_2}$$

$$e ::= \cdots \mid \mathrm{inL}_{\tau_1 + \tau_2}\ e \mid \mathrm{inR}_{\tau_1 + \tau_2}\ e \mid \mathrm{match}\ e\ \mathrm{with}\ e_1 \mid e_2$$
$$v ::= \cdots \mid \mathrm{inL}_{\tau_1 + \tau_2}\ v \mid \mathrm{inR}_{\tau_1 + \tau_2}\ v$$
$$\tau ::= \cdots \mid \tau_1 + \tau_2$$

$$(\mathrm{inL}_{\mathrm{int}+\mathrm{bool}}\ \overset{\mathrm{int}}{5}) : \mathrm{int} + \mathrm{bool}$$
$$(\mathrm{inR}_{\mathrm{int}+\mathrm{bool}}\ \mathrm{true}) : \mathrm{int} + \mathrm{bool}$$

$$\mathrm{match}\ (\mathrm{inL}_{\mathrm{int}+\mathrm{bool}}\ 5)\ \mathrm{with}$$
$$(\lambda x : \mathrm{int}.\ 0) \mid$$
$$(\lambda x : \mathrm{bool}.\ 1)$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathrm{inL}_{\tau_1 + \tau_2}\ e : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathrm{inR}_{\tau_1 + \tau_2}\ e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma \vdash e_1 : \tau_1 \to \tau_3 \quad \Gamma \vdash e_2 : \tau_2 \to \tau_3}{\Gamma \vdash \mathrm{match}\ e\ \mathrm{with}\ e_1 \mid e_2 : \tau_3}$$

$$\lambda x : \mathrm{int}.\ x$$
$$\lambda x : \mathrm{bool}.\ x$$
$$\lambda x : (\mathrm{int} \to \mathrm{bool}).\ x$$

$$(\Lambda \alpha .\ \lambda x : \alpha .\ x)\ (\mathrm{int} \to \mathrm{bool})$$
$$\to \lambda x : (\mathrm{int} \to \mathrm{bool}).\ x$$

$$\Lambda \alpha .\ \Lambda \beta .\ \lambda x : \alpha .\ \lambda y : \beta .\ (x, y)$$